Exercises Work-out

1a) $[A, B_1 B_2] = [A, B_1] B_2 + B_1 [A, B_2]$

$$A B_1 B_2 - B_1 B_2 A = A B_1 B_2 - B_1 \cancel{A B_2} + B_1 \cancel{A B_2} - B_1 B_2 A$$

1b) $[A, B_1 B_2]_+ = [A, B_1]_+ B_2 - B_1 [A, B_2]$

$$A B_1 B_2 + B_1 B_2 A = A B_1 B_2 + B_1 \cancel{A B_2} - B_1 \cancel{A B_2} + B_1 B_2 A$$

1c) Verify relation for $n = 3$

$$[A, B_1 B_2 B_3]_+ = A B_1 B_2 B_3 + B_1 B_2 B_3 A$$

$$\sum_{K=1}^{3} (-1)^{K-1} B_1 \cdots [A, B_K]_+ \cdots B_n$$

$$= [A, B_1]_+ B_2 B_3 - B_1 [A, B_2]_+ B_3 + B_1 B_2 [A, B_3]_+$$

$$= A B_1 B_2 B_3 + B_1 \cancel{A B_2 B_3} - B_1 \cancel{A B_2 B_3} - \cancel{B_1 B_2 A B_3} + \cancel{B_1 B_2 A B_3} + B_1 B_2 B_3 A$$

Then use induction to proof this for all n (odd):

$$[A, B_1 \cdots B_m B_{m+1} B_{m+2}]_+ = [A, B_1 \cdots B_m]_+ B_{m+1} B_{m+2} \qquad \text{(using 1b)}$$

$$- B_1 \cdots B_m [A, B_{m+1} B_{m+2}]$$

$$= [A, B_1 \cdots B_m] B_{m+1} B_{m+2} - B_1 \cdots B_m A B_{m+1} B_{m+2} + B_1 \cdots B_{m+2} A$$

$$= [A, B_1 \cdots B_m]_+ B_{m+1} B_{m+2} - B_1 \cdots B_m [A, B_{m+1}]_+ + B_1 \cdots B_{m+1} A B_{m+2}$$

$$+ B_1 \cdots B_{m+2} A \qquad \left(\text{here we used: } A B_{m+1} = [A, B_{m+1}]_+ - B_{m+1} A\right)$$

$$= [A, B_1 \cdots B_m]_+ B_{m+1} B_{m+2} - B_1 \cdots B_m [A, B_{m+1}]_+ + B_1 \cdots B_{m+1} [A, B_{m+2}]_+$$

Exercise work-out

2) Use Taylor expansion

$$e^{x} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

a) $$(e^{A})^{\dagger} = \left(1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\right)^{\dagger}$$

$$= \left(1 + A^{\dagger} + \frac{1}{2!}(A^{\dagger})^2 + \frac{1}{3!}(A^{\dagger})^3 + \cdots\right)$$

$$= e^{(A^{\dagger})}$$

b) $$B e^{A} B^{-1} = e^{BAB^{-1}}$$

Easiest is to start from the right-hand-side:

$$e^{BAB^{-1}} = 1 + BAB^{-1} + \frac{1}{2!} BAB^{-1}BAB + \frac{1}{3!} BAB^{-1}BAB^{-1}BAB^{-1} + \cdots$$

$$= B\left(1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3\right)B^{-1} = Be^{A}B^{-1}$$

c) $$e^{A+B} = \left(1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\right)\left(1 + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots\right)$$

$$1 + (A+B) + \frac{1}{2}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots = 1 + A + B + \frac{1}{2!}A^2 + \frac{1}{2!}B^2 + AB$$
$$+ \frac{1}{3!}A^3 + \frac{1}{3!}B^3 + \frac{1}{2!}A^2B + \frac{1}{2!}AB^2 + \cdots$$

Comparing both sides order by order reveals that they match provided that

$$\frac{1}{2}(AB + BA) = AB \quad \text{and} \quad \frac{1}{6}(AB^2 + BAB + B^2A + A^2B + ABA + BA^2)$$
$$= \frac{1}{2}(AB^2 + A^2B)$$

this is true if A and B commute.

d) $$\frac{d}{d\lambda} e^{\lambda A} = \frac{d}{d\lambda}\left(1 + \lambda A + \frac{1}{2}\lambda^2 A^2 + \frac{1}{6}\lambda^3 A^3 + \cdots\right)$$

$$= A + \lambda A^2 + \frac{1}{2}\lambda^2 A^3 + \cdots$$

$$= A\left(1 + \lambda A + \frac{1}{2}\lambda^2 A^2 + \cdots\right)$$

$$= A e^{\lambda A}$$

or $$= \left(1 + \lambda A + \frac{1}{2}\lambda^2 A^2 + \cdots\right) A$$

$$= e^{\lambda A} A$$

e) $$e^{-A} B e^{A} = \left(1 - A + \frac{1}{2}A^2 - \frac{1}{6}A^3\right) B \left(1 + A + \frac{1}{2}A^2 + \frac{1}{6}A^3\right)$$

collect terms by order in A:

$$B - AB + BA + \frac{1}{2}A^2B - ABA + \frac{1}{2}BA^2 - \frac{1}{6}A^3B + \frac{1}{2}A^2BA - \frac{1}{2}ABA^2 + \frac{1}{6}BA^3 + \cdots$$

$$= B + [B, A] + \frac{1}{2}[[B, A]A] + \frac{1}{6}[[[B, A], A], A]$$

(verify the last step yourself: write out the commutators for the A^2 and A^3 terms).

3) $$[E_{pq}, E_{rs}] = a^+_p a_q a^+_r a_s - a^+_r a_s a^+_p a_q$$

$$= \delta_{qr} a^+_p a_s - a^+_p a^+_r a_q a_s - \delta_{ps} a^+_r a_q + a^+_r a^+_p a_s a_q$$

(using the elementary commutator $a_q a^+_r = -a^+_r a_q + \delta_{qr}$ to get here).

$$= \delta_{qr} E_{ps} - \delta_{ps} E_{rq}$$

definition of E

(the other two terms cancel each other as we can commute creation and annihilation operators among themselves without getting delta functions)